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ScienceDirect
Journal of Computational Physics 227 (2008) 2195–2197
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Short Note

A proof that a discrete delta function is second-order accurate

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Received 8 August 2007; accepted 5 November 2007
Available online 17 November 2007

Abstract

It is proved that a discrete delta function introduced by Smereka [P. Smereka, The numerical approximation of a delta function with application to level set methods, J. Comput. Phys. 211 (2006) 77–90] gives a second-order accurate quadrature rule for surface integrals using values on a regular background grid. The delta function is found using a technique of Mayo [A. Mayo, The fast solution of Poisson's and the biharmonic equations on irregular regions, SIAM J. Numer. Anal. 21 (1984) 285–299]. It can be expressed naturally using a level set function.

Keywords: Discrete delta function; Level set function; Surface integral

There is considerable interest in designing accurate discrete delta functions for surfaces in a domain covered by a rectangular grid. They can provide quadrature rules for surface integrals using values at regular grid points [2,10–12]. Such a rule is especially useful when the surface is represented by a level set function. In [10] Smereka constructed a discrete delta function as the truncation error in applying the discrete Laplacian to a "Green's function" for the exact delta function on the surface. To find the truncation error, he used the technique of Mayo [7,8] for solving differential equations with interfacial conditions, in which jump conditions are built into the difference operators on a regular grid. (The immersed interface method [3,5], the EJIIM [13,9] and the ghost fluid method [6] are related to Mayo's technique.) Smereka also showed how to express this delta function in terms of a level set function. He conjectured that the resulting quadrature rule for surface integrals is second-order accurate and verified the accuracy in numerical examples. In this note we give a simple proof of this fact.

Suppose Γ is a closed curve in $\mathbb{R}^2$ or a closed surface in $\mathbb{R}^3$, bounding a set which is contained in a rectangular domain Ω. The problem is to design a weight function w^h at grid points on a square grid Ω_h, concentrated near Γ, so that, for any smooth function f defined near the curve Γ in $\mathbb{R}^2$,

DOI of original article: 10.1016/j.jcp.2005.05.005.
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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2007.11.004

$$\int_{\Gamma} f(x)\,\mathrm{d}s(x) = \sum_{ih\in\Omega_h} f(ih)w^h(ih)h^2 + \mathrm{O}(h^2) \tag{1}$$

or near the surface Γ in $\mathbb{R}^3$,

$$\int_{\Gamma} f(x)\,\mathrm{d}S(x) = \sum_{ih\in\Omega_h} f(ih)w^h(ih)h^3 + \mathrm{O}(h^2). \tag{2}$$

Arclength and surface area are special cases. Smereka's w^{h} has support on the grid points ih within distance h of Γ, i.e., $w^h(ih) = 0$ at other points. We will prove that (2) holds, with w^{h} as in [10], assuming Γ is a smooth surface in $\mathbb{R}^3$. The case of a curve in $\mathbb{R}^2$ is entirely similar.

Smereka's procedure is as follows: Let δ_Γ be the distribution, or generalized function, restricting to Γ; that is, for smooth f on Ω,

$$\int_{\Omega} f\delta_\Gamma\,\mathrm{d}x = \int_{\Gamma} f\,\mathrm{d}S. \tag{3}$$

Let g be the solution of

$$\Delta g = \delta_\Gamma \quad \text{in } \Omega, \quad g = 0 \quad \text{on} \quad \partial\Omega. \tag{4}$$

Assuming Γ is smooth, g is piecewise smooth, i.e., smooth and harmonic on each region bounded by Γ, with the jump conditions

$$[g] = 0, \quad [\partial_n g] = 1 \text{ on } \Gamma, \tag{5}$$

where ∂_n is the normal derivative on Γ. In fact g can be thought of as a single layer potential on Γ. Now let Δ_h be the usual second-order discrete Laplacian on Ω_h and let τ^h be the truncation error

$$\Delta_h g = \tau^h \quad \text{on } \Omega_h. \tag{6}$$

Smereka constructs the weights w^{h} from expressions for τ^h, using Mayo's technique [7,8]. At a *regular* grid point $ih \in \Omega_h$, for which the stencil of Δ_h does not cross $\Gamma, \tau^h(ih) = \mathrm{O}(h^2)$ as usual. At an *irregular* grid point, τ^h is larger. It can be found to $\mathrm{O}(h)$ using the jumps in first and second derivatives of g; see (30) in [10]. These in turn can be expressed in derivatives of the normal and tangent vectors to Γ. (See (41), (47) in [10] for $\mathbb{R}^2$ and Section 7.2 for $\mathbb{R}^3$.) Thus τ^h has the form

$$\Delta_h g = \tau^h = w^h + \mathrm{O}_\Gamma(h) + \mathrm{O}(h^2) \quad \text{on } \Omega_h, \tag{7}$$

where w^{h} is known analytically and w^h and $\mathrm{O}_\Gamma(h)$ are nonzero only at the irregular points. The errors are uniform. Smereka shows how to write w^{h} in terms of a level set function; see (45) and Section 7 in [10].

To prove that (2) is valid, we may assume f is nonzero only in a neighborhood of Γ, as well as smooth. We begin by writing

$$\int_{\Gamma} f\,\mathrm{d}S = \int_{\Omega} f\delta_\Gamma\,\mathrm{d}x = \int_{\Omega} f\Delta g\,\mathrm{d}x = \int_{\Omega} g\Delta f\,\mathrm{d}x. \tag{8}$$

(This could be rewritten in an equivalent way using the jump conditions (5) rather than δ_Γ).

Next we replace the last integral by a sum over grid points. We check that

$$\int_{\Omega} g\Delta f\,\mathrm{d}x = \sum_{ih\in\Omega_h} g(ih)(\Delta f)(ih)h^3 + \mathrm{O}(h^2) \tag{9}$$

by comparing the integral over the cell centered at ih with the term in the sum. If the cell intersects Γ, the error in the integrand is $\mathrm{O}(h)$, since g is continuous and has bounded derivative. There are $\mathrm{O}(h^{-2})$ such cells, contributing a total error of $\mathrm{O}(h \cdot h^3 \cdot h^{-2}) = \mathrm{O}(h^2)$. On each remaining cell the error in the integral is $\mathrm{O}(h^2 \cdot h^3)$, since g and Δf are C^2. The total error for these cells is $\mathrm{O}(h^2 \cdot h^3 \cdot h^{-3}) = \mathrm{O}(h^2)$ and the claim (9) is verified.

We now have

$$\int_{\Gamma} f\,\mathrm{d}S = \sum_{\Omega_h} g\Delta f h^3 + \mathrm{O}(h^2) = \sum_{\Omega_h} g\Delta_h f h^3 + \mathrm{O}(h^2) \tag{10}$$

since $\Delta_h f = \Delta f + \mathrm{O}(h^2)$. We can sum by parts and use (7) to obtain

$$\sum_{\Omega_h} g\Delta_h f h^3 = \sum_{\Omega_h} (\Delta_h g) f h^3 = \sum_{\Omega_h} \Big(w^h + \mathrm{O}_\Gamma(h) + \mathrm{O}(h^2) \Big) f h^3. \tag{11}$$

The $\mathrm{O}_\Gamma(h)$ error contributes a term of order $h \cdot h^3 \cdot h^{-2} = h^2$ and thus is negligible, as is the other error inside. Combining (10) and (11), we arrive at the conclusion (2).

The fact that the integral is accurate to $\mathrm{O}(h^2)$ although $\tau^h = \mathrm{O}(h)$ on the irregular points is related to a gain in accuracy that has long been noted for solutions of elliptic problems using the methods of [3–5,7,8,13]. Proofs of this phenomenon have been given in [1,4,9] and elsewhere. Closely related to the Green's function g solving (4) is the discrete version g^h which solves

$$\Delta_h g^h = w^h \quad \text{in } \Omega_h, \quad g = 0 \quad \text{on } \partial\Omega_h. \tag{12}$$

In fact $g^h - g = \mathrm{O}(h^2)$ uniformly; this follows from analytical results in [1,9].

Acknowledgment

This material is based upon work supported by the National Science Foundation under grant DMS-0404765.

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